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# The Terwilliger algebra of a distance-regular graph of negative type ☆

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**Abstract**

Let $\Gamma$ denote a distance-regular graph with diameter $D \geqslant 3$. Assume $\Gamma$ has classical parameters $(D, b, \alpha, \beta)$ with $b < -1$. Let $X$ denote the vertex set of $\Gamma$ and let $A \in \mathrm{Mat}_X(\mathbb{C})$ denote the adjacency matrix of $\Gamma$. Fix $x \in X$ and let $A^* \in \mathrm{Mat}_X(\mathbb{C})$ denote the corresponding dual adjacency matrix. Let $T$ denote the subalgebra of $\mathrm{Mat}_X(\mathbb{C})$ generated by $A, A^*$. We call $T$ the *Terwilliger algebra* of $\Gamma$ with respect to $x$. We show that up to isomorphism there exist exactly two irreducible $T$-modules with endpoint 1; their dimensions are $D$ and $2D-2$. For these $T$-modules we display a basis consisting of eigenvectors for $A^*$, and for each basis we give the action of $A$.


*AMS classification:* 05E30

*Keywords:* Distance-regular graph; Negative type; Terwilliger algebra

## 1. Introduction

Let $\Gamma$ denote a $Q$-polynomial distance-regular graph with diameter $D \geqslant 3$ and intersection numbers $a_i, b_i, c_i$ (see Section 2 for formal definitions). We recall the Terwilliger algebra of $\Gamma$. Let $X$ denote the vertex set of $\Gamma$ and let $A \in \mathrm{Mat}_X(\mathbb{C})$ denote the adjacency matrix of $\Gamma$. Fix a "base vertex" $x \in X$ and let $A^* \in \mathrm{Mat}_X(\mathbb{C})$ denote the corresponding dual adjacency matrix. Let

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$T = T(x)$ denote the subalgebra of $\mathrm{Mat}_X(\mathbb{C})$ generated by $A$, $A^*$. The algebra $T$ is called the *Terwilliger algebra* of $\Gamma$ with respect to $x$ [28]. $T$ is closed under the conjugate–transpose map so $T$ is semi-simple [28, Lemma 3.4(i)]. Therefore, each $T$-module is a direct sum of irreducible $T$-modules. Describing the irreducible $T$-modules is an active area of research [3–17,21,26,28,31].

In this description there is an important parameter called the *endpoint* which we now recall. Let $W$ denote an irreducible $T$-module. By the *endpoint* of $W$ we mean $\min\{i \mid 0 \leqslant i \leqslant D,\ E_i^* W \neq 0\}$, where $E_i^* \in \mathrm{Mat}_X(\mathbb{C})$ is the projection onto the $i$th subconstituent of $\Gamma$ with respect to $x$ [28, p. 378]. There exists a unique irreducible $T$-module with endpoint 0 [12, Proposition 8.4]; for a detailed description see [7,12].

Consider now the irreducible $T$-modules with endpoint 1. If $\Gamma$ is bipartite, then these $T$-modules are described in [7,8]. If $\Gamma$ is nonbipartite with $a_1 = 0$, then these $T$-modules are described in [4,21]. For the rest of this Introduction assume $a_1 \neq 0$. Assume further that $\Gamma$ is of negative type and not a near polygon. In [22], we described the combinatorial structure of $\Gamma$. In the present paper, we use this description to obtain the irreducible $T$-modules that have endpoint 1. To summarize our results we note the following. Let $W$ denote an irreducible $T$-module with endpoint 1. Observe that $E_1^* W$ is a one-dimensional eigenspace for $E_1^* A E_1^*$ [16, Theorem 2.2]. The corresponding eigenvalue is called the *local eigenvalue* of $W$. We show that up to isomorphism there exist exactly two irreducible $T$-modules with endpoint 1. The first one has dimension $D$ and local eigenvalue $-1$. The second one has dimension $2D - 2$ and local eigenvalue $a_1$. For these modules we display a basis consisting of eigenvectors for $A^*$, and for each basis we give the action of $A$. At present there is no classification of graphs that satisfy our assumptions; see [22, Section 6] for a summary of what is known.

## 2. Preliminaries

In this section, we review some definitions and basic results concerning distance-regular graphs. See the book of Brouwer, Cohen and Neumaier [2] for more background information.

Let $\mathbb{C}$ denote the complex number field and let $X$ denote a nonempty finite set. Let $\mathrm{Mat}_X(\mathbb{C})$ denote the $\mathbb{C}$-algebra consisting of all matrices whose rows and columns are indexed by $X$ and whose entries are in $\mathbb{C}$. Let $V = \mathbb{C}^X$ denote the vector space over $\mathbb{C}$ consisting of column vectors whose coordinates are indexed by $X$ and whose entries are in $\mathbb{C}$. We observe $\mathrm{Mat}_X(\mathbb{C})$ acts on $V$ by left multiplication. We call $V$ the *standard module*. We endow $V$ with the Hermitean inner product $\langle\, ,\, \rangle$ that satisfies $\langle u, v\rangle = u^{\mathrm{t}}\bar{v}$ for $u, v \in V$, where $t$ denotes transpose and $\bar{}$ denotes complex conjugation. For $y \in X$ let $\hat{y}$ denote the element of $V$ with a 1 in the $y$ coordinate and 0 in all other coordinates. We observe $\{\hat{y} \mid y \in X\}$ is an orthonormal basis for $V$. The following will be useful: for each $B \in \mathrm{Mat}_X(\mathbb{C})$ we have

$$\langle u, Bv\rangle = \langle \overline{B}^{\mathrm{t}} u, v\rangle \quad (u, v \in V). \tag{1}$$

Let $\Gamma = (X, R)$ denote a finite, undirected, connected graph, without loops or multiple edges, with vertex set $X$ and edge set $R$. Let $\partial$ denote the path-length distance function for $\Gamma$, and set $D := \max\{\partial(x, y) \mid x, y \in X\}$. We call $D$ the *diameter* of $\Gamma$. For a vertex $x \in X$ and an integer $i$ let $\Gamma_i(x)$ denote the set of vertices at distance $i$ from $x$. We abbreviate $\Gamma(x) = \Gamma_1(x)$. For an integer $k \geqslant 0$ we say $\Gamma$ is *regular with valency* $k$ whenever $|\Gamma(x)| = k$ for all $x \in X$. We say $\Gamma$ is *distance-regular* whenever for all integers $h, i, j$ $(0 \leqslant h, i, j \leqslant D)$ and for all vertices $x, y \in X$ with $\partial(x, y) = h$, the number

$$p^h_{ij} = |\Gamma_i(x) \cap \Gamma_j(y)|$$

is independent of $x$ and $y$. The $p^h_{ij}$ are called the *intersection numbers* of $\Gamma$.

For the rest of this paper we assume $\Gamma$ is distance-regular with diameter $D \geqslant 3$. Note that $p^h_{ij} = p^h_{ji}$ for $0 \leqslant h, i, j \leqslant D$. For convenience set $c_i := p^i_{1,i-1}$ $(1 \leqslant i \leqslant D)$, $a_i := p^i_{1i}$ $(0 \leqslant i \leqslant D)$, $b_i := p^i_{1,i+1}$ $(0 \leqslant i \leqslant D-1)$, $k_i := p^0_{ii}$ $(0 \leqslant i \leqslant D)$, and $c_0 = b_D = 0$. By the triangle inequality the following hold for $0 \leqslant h, i, j \leqslant D$: (i) $p^h_{ij} = 0$ if one of $h, i, j$ is greater than the sum of the other two; (ii) $p^h_{ij} \neq 0$ if one of $h, i, j$ equals the sum of the other two. In particular $c_i \neq 0$ for $1 \leqslant i \leqslant D$ and $b_i \neq 0$ for $0 \leqslant i \leqslant D-1$. We observe that $\Gamma$ is regular with valency $k = k_1 = b_0$ and that

$$c_i + a_i + b_i = k \quad (0 \leqslant i \leqslant D). \tag{2}$$

Note that $k_i = |\Gamma_i(x)|$ for $x \in X$ and $0 \leqslant i \leqslant D$. By [2, p. 127]

$$k_i = \frac{b_0 b_1 \cdots b_{i-1}}{c_1 c_2 \cdots c_i} \quad (0 \leqslant i \leqslant D). \tag{3}$$

We recall the Bose–Mesner algebra of $\Gamma$. For $0 \leqslant i \leqslant D$ let $A_i$ denote the matrix in $\mathrm{Mat}_X(\mathbb{C})$ with $(x, y)$-entry

$$(A_i)_{xy} = \begin{cases} 1 & \text{if } \partial(x,y) = i, \\ 0 & \text{if } \partial(x,y) \neq i, \end{cases} \quad (x, y \in X). \tag{4}$$

We call $A_i$ the $i$th *distance matrix* of $\Gamma$. We abbreviate $A := A_1$ and call this the *adjacency matrix* of $\Gamma$. We observe (ai) $A_0 = I$; (aii) $\sum_{i=0}^D A_i = J$; (aiii) $\overline{A_i} = A_i$ $(0 \leqslant i \leqslant D)$; (aiv) $A_i^{\mathrm{t}} = A_i$ $(0 \leqslant i \leqslant D)$; (av) $A_i A_j = \sum_{h=0}^D p^h_{ij} A_h$ $(0 \leqslant i, j \leqslant D)$, where $I$ (resp. $J$) denotes the identity matrix (resp. all 1's matrix) in $\mathrm{Mat}_X(\mathbb{C})$. Using these facts we find $A_0, A_1, \ldots, A_D$ is a basis for a commutative subalgebra $M$ of $\mathrm{Mat}_X(\mathbb{C})$. We call $M$ the *Bose–Mesner algebra* of $\Gamma$. It turns out that $A$ generates $M$ [1, p. 190]. By [2, p. 45], $M$ has a second basis $E_0, E_1, \ldots, E_D$ such that (ei) $E_0 = |X|^{-1} J$; (eii) $\sum_{i=0}^D E_i = I$; (eiii) $\overline{E_i} = E_i$ $(0 \leqslant i \leqslant D)$; (eiv) $E_i^{\mathrm{t}} = E_i$ $(0 \leqslant i \leqslant D)$; (ev) $E_i E_j = \delta_{ij} E_i$ $(0 \leqslant i, j \leqslant D)$. We call $E_0, E_1, \ldots, E_D$ the *primitive idempotents* of $\Gamma$.

We now recall the Krein parameters. Let $\circ$ denote the entrywise product in $\mathrm{Mat}_X(\mathbb{C})$. Observe $A_i \circ A_j = \delta_{ij} A_i$ for $0 \leqslant i, j \leqslant D$, so $M$ is closed under $\circ$. Thus there exist complex scalars $q^h_{ij}$ $(0 \leqslant h, i, j \leqslant D)$ such that

$$E_i \circ E_j = |X|^{-1} \sum_{h=0}^{D} q^h_{ij} E_h \quad (0 \leqslant i, j \leqslant D).$$

By [2, Proposition 4.1.5], $q^h_{ij}$ is real and nonnegative for $0 \leqslant h, i, j \leqslant D$. The $q^h_{ij}$ are called the *Krein parameters of* $\Gamma$. The graph $\Gamma$ is said to be *Q-polynomial* (with respect to the given ordering $E_0, E_1, \ldots, E_D$ of the primitive idempotents) whenever for $0 \leqslant h, i, j \leqslant D$, $q^h_{ij} = 0$ (resp. $q^h_{ij} \neq 0$) whenever one of $h, i, j$ is greater than (resp. equal to) the sum of the other two. For the rest of this section assume $\Gamma$ is $Q$-polynomial with respect to $E_0, E_1, \ldots, E_D$.

We now recall the dual idempotents of $\Gamma$. To do this fix a vertex $x \in X$. We view $x$ as a "base vertex". For $0 \leqslant i \leqslant D$ let $E_i^* = E_i^*(x)$ denote the diagonal matrix in $\mathrm{Mat}_X(\mathbb{C})$ with $(y, y)$-entry

$$(E_i^*)_{yy} = \begin{cases} 1 & \text{if } \partial(x, y) = i, \\ 0 & \text{if } \partial(x, y) \neq i, \end{cases} \quad (y \in X). \tag{5}$$

We call $E_i^*$ the $i$th *dual idempotent* of $\Gamma$ with respect to $x$ [28, p. 378]. We observe (i) $\sum_{i=0}^{D} E_i^* = I$; (ii) $\overline{E_i^*} = E_i^*$ $(0 \leqslant i \leqslant D)$; (iii) $E_i^{*t} = E_i^*$ $(0 \leqslant i \leqslant D)$; (iv) $E_i^* E_j^* = \delta_{ij} E_i^*$ $(0 \leqslant i, j \leqslant D)$. By these facts $E_0^*, E_1^*, \ldots, E_D^*$ form a basis for a commutative subalgebra $M^* = M^*(x)$ of $\mathrm{Mat}_X(\mathbb{C})$. We call $M^*$ the *dual Bose–Mesner algebra* of $\Gamma$ with respect to $x$ [28, p. 378]. For $0 \leqslant i \leqslant D$ we have

$$E_i^* V = \mathrm{Span}\{\hat{y} | y \in X, \partial(x, y) = i\},$$

so $\dim E_i^* V = k_i$. We call $E_i^* V$ the $i$th *subconstituent of* $\Gamma$ with respect to $x$. Note that

$$V = E_0^* V + E_1^* V + \cdots + E_D^* V \quad \text{(orthogonal direct sum)}. \tag{6}$$

Moreover, $E_i^*$ is the projection from $V$ onto $E_i^* V$ for $0 \leqslant i \leqslant D$. Let $A^* = A^*(x)$ denote the diagonal matrix in $\mathrm{Mat}_X(\mathbb{C})$ with $(y, y)$-entry

$$A_{yy}^* = |X| E_{xy} \quad (y \in X),$$

where $E = E_1$. We call $A^*$ the *dual adjacency matrix of* $\Gamma$ with respect to $x$. By [28, Lemma 3.11(ii)] $A^*$ generates $M^*$.

We recall the Terwilliger algebra of $\Gamma$. Let $T = T(x)$ denote the subalgebra of $\mathrm{Mat}_X(\mathbb{C})$ generated by $M$, $M^*$. We call $T$ the *Terwilliger algebra of* $\Gamma$ with respect to $x$ [28, Definition 3.3]. Recall $M$ (resp. $M^*$) is generated by $A$ (resp. $A^*$) so $T$ is generated by $A$, $A^*$. We observe $T$ has finite dimension. By construction $T$ is closed under the conjugate–transpose map so $T$ is semi-simple [28, Lemma 3.4(i)].

By a $T$-*module* we mean a subspace $W$ of $V$ such that $BW \subseteq W$ for all $B \in T$. Let $W$ denote a $T$-module. Then $W$ is said to be *irreducible* whenever $W$ is nonzero and $W$ contains no $T$-modules other than 0 and $W$. Assume $W$ is irreducible. Then $A$ and $A^*$ act on $W$ as a tridiagonal pair [17, Example 1.4]. We refer the reader to [17–20,24,25] and the references therein for background on tridiagonal pairs.

By [14, Corollary 6.2] any $T$-module is an orthogonal direct sum of irreducible $T$-modules. In particular the standard module $V$ is an orthogonal direct sum of irreducible $T$-modules. Let $W$, $W'$ denote $T$-modules. By an *isomorphism of* $T$-*modules* from $W$ to $W'$ we mean an isomorphism of vector spaces $\sigma : W \to W'$ such that $(\sigma B - B\sigma)W = 0$ for all $B \in T$. The $T$-modules $W$, $W'$ are said to be *isomorphic* whenever there exists an isomorphism of $T$-modules from $W$ to $W'$. By [7, Lemma 3.3] any two nonisomorphic irreducible $T$-modules are orthogonal. Let $W$ denote an irreducible $T$-module. By [28, Lemma 3.4(iii)] $W$ is an orthogonal direct sum of the nonvanishing spaces among $E_0^* W, E_1^* W, \ldots, E_D^* W$. By the *endpoint* of $W$ we mean $\min\{i | 0 \leqslant i \leqslant D, E_i^* W \neq 0\}$. By the *diameter* of $W$ we mean $|\{i | 0 \leqslant i \leqslant D, E_i^* W \neq 0\}| - 1$.

By [12, Propositions 8.3, 8.4] $M\hat{x}$ is the unique irreducible $T$-module with endpoint 0 and the unique irreducible $T$-module with diameter $D$. Moreover, $M\hat{x}$ is the unique irreducible $T$-module on which $E_0$ does not vanish. We call $M\hat{x}$ the *primary module*.

We finish this section with some comments on local eigenvalues. Let $\Delta = \Delta(x)$ denote the vertex-subgraph of $\Gamma$ induced on the set of vertices in $X$ adjacent $x$, and let $\breve{A}$ denote the adjacency matrix of $\Delta$. By the *local eigenvalues of* $\Gamma$ we mean the eigenvalues of $\breve{A}$. Note that the local eigenvalues of $\Gamma$ are precisely the eigenvalues of $E_1^* A E_1^*$ on $E_1^* V$.

Let $W$ denote an irreducible $T$-module with endpoint 1. By [16, Theorem 2.2] $E_1^* W$ is a one-dimensional eigenspace for $E_1^* A E_1^*$; we call the corresponding eigenvalue the *local eigenvalue* of $W$.

## 3. Distance-regular graphs of negative type

In this section, we recall what it means for $\Gamma$ to have classical parameters and negative type. The graph $\Gamma$ is said to have *classical parameters* $(D, b, \alpha, \beta)$ whenever the intersection numbers of $\Gamma$ satisfy

$$c_i = \begin{bmatrix} i \\ 1 \end{bmatrix} \left(1 + \alpha \begin{bmatrix} i-1 \\ 1 \end{bmatrix}\right) \quad (1 \leqslant i \leqslant D),$$

$$b_i = \left(\begin{bmatrix} D \\ 1 \end{bmatrix} - \begin{bmatrix} i \\ 1 \end{bmatrix}\right) \left(\beta - \alpha \begin{bmatrix} i \\ 1 \end{bmatrix}\right) \quad (0 \leqslant i \leqslant D-1),$$

where

$$\begin{bmatrix} j \\ 1 \end{bmatrix} := 1 + b + b^2 + \cdots + b^{j-1}.$$

In this case $b$ is an integer and $b \notin \{0, -1\}$. If $\Gamma$ has classical parameters then $\Gamma$ is $Q$-polynomial [2, Corollary 8.4.2]. We say that $\Gamma$ has *negative type* whenever $\Gamma$ has classical parameters $(D, b, \alpha, \beta)$ such that $b < -1$.

We now recall kites and parallelograms. Fix an integer $i$ $(2 \leqslant i \leqslant D)$. By a *kite of length* $i$ (or $i$*-kite*) in $\Gamma$ we mean a 4-tuple $uvwz$ of vertices of $\Gamma$ such that $u, v, w$ are mutually adjacent, and $\partial(u, z) = i$, $\partial(v, z) = \partial(w, z) = i - 1$. By a *parallelogram of length* $i$ (or $i$*-parallelogram*) in $\Gamma$ we mean a 4-tuple $uvwz$ of vertices of $\Gamma$ such that $\partial(u, v) = \partial(w, z) = 1$, $\partial(u, z) = i$, and $\partial(v, z) = \partial(u, w) = \partial(v, w) = i - 1$. By [27, Theorem 2.12, 22, Theorem 4.2], if $\Gamma$ has negative type then $\Gamma$ has no parallelograms or kites of any length.

We now recall the near polygons. The graph $\Gamma$ is called a *near polygon* whenever $a_i = a_1 c_i$ for $1 \leqslant i \leqslant D - 1$ and $\Gamma$ has no 2-kite [23]. From now on we adopt the following notational convention.

**Notation 3.1.** Let $\Gamma = (X, R)$ denote a distance-regular graph with classical parameters $(D, b, \alpha, \beta)$, $D \geqslant 3$, with valency $k$ and $a_1 \neq 0$. Assume that $\Gamma$ is of negative type and $\Gamma$ is not a near polygon. Let $A_0, A_1, \ldots, A_D$ denote the distance matrices of $\Gamma$, and let $V$ denote the standard module of $\Gamma$. We fix $x \in X$ and let $E_i^* = E_i^*(x)$ $(0 \leqslant i \leqslant D)$, and $T = T(x)$ denote the corresponding dual idempotents and Terwilliger algebra, respectively.

The following result is an immediate consequence of [22, Lemmas 6.4 and 6.5].

**Corollary 3.2.** *With reference to Notation* 3.1 *we have* $a_i > a_1 c_i$ *for* $2 \leqslant i \leqslant D$.

## 4. The sets $D_j^i$

With reference to Notation 3.1, in this section we define certain subsets $D_j^i$ of $X$ and explore their properties.

**Definition 4.1.** With reference to Notation 3.1 fix $z \in \Gamma(x)$. For all integers $i, j$ we define $D^i_j = D^i_j(x, z)$ by

$$D^i_j = \Gamma_i(x) \cap \Gamma_j(z).$$

We observe $D^i_j = \emptyset$ unless $0 \leqslant i, j \leqslant D$.

**Lemma 4.2.** *With reference to Notation* 3.1 *and Definition* 4.1 *the following* (i), (ii) *hold for* $0 \leqslant i, j \leqslant D$:

(i) $|D^i_j| = p^1_{ij}$.
(ii) $D^i_j = \emptyset$ *if and only if* $p^1_{ij} = 0$.

**Proof.** (i) Immediate from the definition of $p^1_{ij}$ and $D^i_j$.
(ii) Immediate from (i) above. □

**Lemma 4.3** [2, p. 134]. *With reference to Notation* 3.1 *the following* (i), (ii) *hold*:

(i) $p^1_{i-1,i} = p^1_{i,i-1} = c_i k_i k^{-1}$ $(1 \leqslant i \leqslant D)$.
(ii) $p^1_{ii} = a_i k_i k^{-1}$ $(0 \leqslant i \leqslant D)$.

**Lemma 4.4.** *With reference to Notation* 3.1 *the following* (i)–(iii) *hold*:

(i) $p^1_{i-1,i} \neq 0$, $p^1_{i,i-1} \neq 0$ $(1 \leqslant i \leqslant D)$.
(ii) $p^1_{00} = 0$, $p^1_{ii} \neq 0$ $(1 \leqslant i \leqslant D)$.
(iii) $p^1_{ij} = 0$ *if* $|i - j| \notin \{0, 1\}$ $(0 \leqslant i, j \leqslant D)$.

**Proof.** (i) Immediate from Lemma 4.3(i).
(ii) It is clear that $p^1_{00} = 0$ and $p^1_{11} = a_1 \neq 0$. Assume $2 \leqslant i \leqslant D$. By Corollary 3.2, we find $a_i \neq 0$ so $p^1_{ii} \neq 0$ in view of Lemma 4.3(ii).
(iii) Immediate from the triangle inequality. □

**Lemma 4.5.** *With reference to Notation* 3.1 *and Definition* 4.1 *the following* (i)–(iii) *hold*:

(i) $\partial(u, y) = 1$ *for all distinct* $u, y \in D^1_1$.
(ii) *There are no edges between* $D^{i-1}_i \cup D^i_{i-1}$ *and* $D^{i-1}_{i-1}$ *for* $2 \leqslant i \leqslant D$.
(iii) *For* $1 \leqslant i \leqslant D$ *we have* $\partial(u, y) = i$ *for all* $u \in D^1_1$ *and all* $y \in D^{i-1}_i \cup D^i_{i-1}$.

**Proof.** (i) If $u, y$ are not adjacent, then $yxzu$ is a 2-kite, a contradiction.
(ii) There does not exist adjacent vertices $v, w$ with $v \in D^i_{i-1}$ and $w \in D^{i-1}_{i-1}$; otherwise $xzwv$ is an $i$-parallelogram, a contradiction. A similar argument shows that there does not exist adjacent vertices $v, w$ with $v \in D^{i-1}_i$ and $w \in D^{i-1}_{i-1}$.
(iii) If $i = 1$ then the result is clear. Assume $2 \leqslant i \leqslant D$. By the triangle inequality we find $\partial(u, y) \in \{i - 1, i\}$. But $\partial(u, y) \geqslant i$ by (ii) above, and the result follows. □

We end this section with two remarks on the local eigenvalues.

**Corollary 4.6.** *With reference to Notation 3.1, let $\Delta = \Delta(x)$ denote the vertex-subgraph of $\Gamma$ induced on the set of vertices in $X$ adjacent $x$. Then the following* (i), (ii) *hold*:

(i) *$\Delta$ is a disjoint union of $k(a_1+1)^{-1}$ cliques, each consisting of $a_1+1$ vertices.*
(ii) *The local eigenvalues of $\Gamma$ are $a_1$ with multiplicity $k(a_1+1)^{-1}$, and $-1$ with multiplicity $ka_1(a_1+1)^{-1}$.*

**Proof.** (i) Immediate from Lemma 4.5(i) and (ii).
(ii) Immediate from (i) above. □

**Corollary 4.7.** *With reference to Notation 3.1, let $W$ denote an irreducible $T$-module with endpoint 1. Then the local eigenvalue of $W$ is $a_1$ or $-1$.*

**Proof.** The local eigenvalue of $W$ is a local eigenvalue of $\Gamma$. The result now follows from Corollary 4.6(ii). □

## 5. The sets $D_i^i(0)$ and $D_i^i(1)$

With reference to Notation 3.1, in this section we define certain subsets $D_i^i(0)$ and $D_i^i(1)$ of $X$ and explore their properties.

**Lemma 5.1.** *With reference to Notation 3.1 and Definition 4.1, for $1 \leqslant i \leqslant D$ and $y \in D_i^i$ we have $|\Gamma_{i-1}(y) \cap D_1^1| \leqslant 1$.*

**Proof.** Assume that $|\Gamma_{i-1}(y) \cap D_1^1| \geqslant 2$ and pick distinct $u, v \in \Gamma_{i-1}(y) \cap D_1^1$. Then $xuvy$ is an $i$-kite, a contradiction. □

**Definition 5.2.** With reference to Notation 3.1 and Definition 4.1, for an integer $i$ and $j \in \{0, 1\}$ define a set $D_i^i(j) = D_i^i(j)(x, z)$ by

$$D_i^i(j) = \{y \in D_i^i \mid |\Gamma_{i-1}(y) \cap D_1^1| = j\}.$$

We observe $D_i^i(j) = \emptyset$ unless $1 \leqslant i \leqslant D$. By Lemma 5.1 $D_i^i$ is the disjoint union of $D_i^i(1)$ and $D_i^i(0)$.

**Lemma 5.3** [22, Lemma 6.4]. *With reference to Notation 3.1 and Definition 5.2 the following* (i), (ii) *hold for $1 \leqslant i \leqslant D$*:

(i) $|D_i^i(1)| = a_1 c_i k_i k^{-1}$.
(ii) $|D_i^i(0)| = (a_i - a_1 c_i) k_i k^{-1}$.

**Lemma 5.4.** *With reference to Notation 3.1 and Definition 5.2 the following* (i), (ii) *hold for $1 \leqslant i \leqslant D$*:

(i) *$D_i^i(1) \neq \emptyset$ for $1 \leqslant i \leqslant D$.*
(ii) *$D_i^i(0) \neq \emptyset$ for $2 \leqslant i \leqslant D$ and $D_1^1(0) = \emptyset$.*

**Proof.** Combine Corollary 3.2 and Lemma 5.3. □

**Lemma 5.5** [22, Lemma 4.4(i)]. *With reference to Notation 3.1, Definitions 4.1 and 5.2, for $2 \leqslant i \leqslant D$ we have $\partial(u, y) = i$ for all $u \in D_1^1$ and all $y \in D_i^i(0)$.*

**Lemma 5.6** [22, Sections 5 and 6]. *With reference to Notation 3.1, Definitions 4.1 and 5.2 the following* (i)–(iii) *hold*:

(i) *For $1 \leqslant i \leqslant D$, each vertex in $D_{i-1}^i$ (resp. $D_i^{i-1}$) is adjacent to*

| | | |
|---|---|---|
| *precisely* | $c_{i-1}$ | *vertices in* $D_{i-2}^{i-1}$ (*resp.* $D_{i-1}^{i-2}$), |
| *precisely* | $c_i - c_{i-1}$ | *vertices in* $D_i^{i-1}$ (*resp.* $D_{i-1}^i$), |
| *precisely* | $a_{i-1}$ | *vertices in* $D_{i-1}^i$ (*resp.* $D_i^{i-1}$), |
| *precisely* | $b_i$ | *vertices in* $D_i^{i+1}$ (*resp.* $D_{i+1}^i$), |
| *precisely* | $a_1(c_i - c_{i-1})$ | *vertices in* $D_i^i(1)$, |
| *precisely* | $a_i - a_{i-1} - a_1(c_i - c_{i-1})$ | *vertices in* $D_i^i(0)$, |

*and no other vertices in $X$.*

(ii) *For $2 \leqslant i \leqslant D$, each vertex in $D_i^i(0)$ is adjacent to*

| | | |
|---|---|---|
| *precisely* | $c_i(b^{i-2} - 1)(b^i - 1)^{-1}$ | *vertices in* $D_{i-1}^{i-1}(0)$, |
| *precisely* | $a_1 c_i(b^i - b^{i-2})(b^i - 1)^{-1}$ | *vertices in* $D_i^i(1)$, |
| *precisely* | $c_i(b^i - b^{i-2})(b^i - 1)^{-1}$ | *vertices in* $D_{i-1}^i$, |
| *precisely* | $c_i(b^i - b^{i-2})(b^i - 1)^{-1}$ | *vertices in* $D_i^{i-1}$, |
| *precisely* | $b_i$ | *vertices in* $D_{i+1}^{i+1}(0)$, |
| *precisely* | $a_i - c_i(a_1 + 1)(b^i - b^{i-2})(b^i - 1)^{-1}$ | *vertices in* $D_i^i(0)$, |

*and no other vertices in $X$.*

(iii) *For $1 \leqslant i \leqslant D$, each vertex in $D_i^i(1)$ is adjacent to*

| | | |
|---|---|---|
| *precisely* | $c_{i-1}$ | *vertices in* $D_{i-1}^{i-1}(1)$, |
| *precisely* | $(a_1 - 1)(c_i - c_{i-1}) + a_{i-1}$ | *vertices in* $D_i^i(1)$, |
| *precisely* | $c_i - c_{i-1}$ | *vertices in* $D_{i-1}^i$, |
| *precisely* | $c_i - c_{i-1}$ | *vertices in* $D_i^{i-1}$, |
| *precisely* | $b_i$ | *vertices in* $D_{i+1}^{i+1}(1)$, |
| *precisely* | $a_i - a_{i-1} - a_1(c_i - c_{i-1})$ | *vertices in* $D_i^i(0)$, |

*and no other vertices in $X$.*

## 6. Some products in $T$

With reference to Notation 3.1, in this section we evaluate several products in $T$ which we shall need later.

**Lemma 6.1.** *With reference to Notation 3.1, for $0 \leqslant h, i, j \leqslant D$ and $y, z \in X$ the $(y, z)$-entry of $E_h^* A_i E_j^*$ is 1 if $\partial(x, y) = h$, $\partial(y, z) = i$, $\partial(x, z) = j$, and 0 otherwise.*

**Proof.** Compute the $(y, z)$-entry of $E_h^* A_i E_j^*$ by matrix multiplication and simplify the result using (4) and (5). □

**Corollary 6.2** [28, Lemma 3.2]. *With reference to Notation 3.1*

$$E_h^* A_i E_j^* = 0 \quad \text{if and only if } p_{ij}^h = 0 \quad (0 \leqslant h, i, j \leqslant D).$$

**Proof.** Immediate from Lemma 6.1. □

**Corollary 6.3.** *With reference to Notation 3.1 and Definition 4.1 for $0 \leqslant i, j \leqslant D$ and $y \in X$ the $(y, z)$-entry of $E_i^* A_j E_1^*$ is 1 if $y \in D_j^i$, and 0 otherwise.*

**Proof.** Immediate from Lemma 6.1. □

**Corollary 6.4.** *With reference to Notation 3.1 the following* (i), (ii) *hold*:

(i) $E_i^* A_{i-1} E_1^* + E_i^* A_i E_1^* + E_i^* A_{i+1} E_1^* = E_i^* J E_1^*$ *for* $1 \leqslant i \leqslant D-1$.
(ii) $E_D^* A_{D-1} E_1^* + E_D^* A_D E_1^* = E_D^* J E_1^*$.

**Proof.** For each equation evaluate the right-hand side using assertion (aii) below line (4), and simplify the result using Corollary 6.2 and assertion (i) above line (2). □

**Lemma 6.5.** *With reference to Notation 3.1 for $0 \leqslant h, i, j, r, s \leqslant D$ and $y, z \in X$ the $(y, z)$-entry of $E_h^* A_r E_i^* A_s E_j^*$ is $|\Gamma_i(x) \cap \Gamma_r(y) \cap \Gamma_s(z)|$ if $\partial(x, y) = h$, $\partial(x, z) = j$, and 0 otherwise.*

**Proof.** Compute the $(y, z)$-entry of $E_h^* A_r E_i^* A_s E_j^*$ by matrix multiplication and simplify the result using (4) and (5). □

**Corollary 6.6.** *With reference to Notation 3.1 and Definition 5.2, for $1 \leqslant i \leqslant D$ and $y \in X$ the following* (i), (ii) *hold*:

(i) *The $(y, z)$-entry of $E_i^* A_{i-1} E_1^* A E_1^*$ is 1 if $y \in D_i^i(1)$, and 0 otherwise.*
(ii) *The $(y, z)$-entry of $E_i^*(A_i - A_{i-1} E_1^* A) E_1^*$ is 1 if $y \in D_i^i(0)$, and 0 otherwise.*

**Proof.** (i) Immediate from Lemmas 4.5(iii), 5.5 and 6.5.

(ii) By Corollary 6.3 the $(y, z)$-entry of $E_i^* A_i E_1^*$ is 1 if $y \in D_i^i$, and 0 otherwise. The result now follows from (i) above and since $D_i^i$ is the disjoint union of $D_i^i(0)$ and $D_i^i(1)$. □

## 7. The matrices $L$, $F$, $R$

With reference to Notation 3.1, in this section we recall the matrices $L$, $F$, $R$ and use them to interpret Theorem 5.6.

**Definition 7.1.** With reference to Notation 3.1 we define matrices $L = L(x)$, $F = F(x)$, $R = R(x)$ by

$$L = \sum_{h=1}^{D} E_{h-1}^* A E_h^*, \quad F = \sum_{h=0}^{D} E_h^* A E_h^*, \quad R = \sum_{h=0}^{D-1} E_{h+1}^* A E_h^*.$$

Note that $A = L + F + R$ [7, Lemma 4.4]. We call $L$, $F$, and $R$ the *lowering matrix*, the *flat matrix*, and the *raising matrix of* $\Gamma$ *with respect to* $x$.

**Lemma 7.2.** *With reference to Notation* 3.1 *and Definition* 7.1 *the following* (i)–(iii) *hold*:

(i) $LE_1^* = E_0^* A E_1^*$.
(ii) *For* $2 \leqslant i \leqslant D$

$$LE_i^* A_{i-1} E_1^* = b_{i-1} E_{i-1}^* A_{i-2} E_1^* + (c_i - c_{i-1}) E_{i-1}^* A_i E_1^*.$$

(iii) *For* $1 \leqslant i \leqslant D-1$

$$LE_i^* A_{i+1} E_1^* = b_i E_{i-1}^* A_i E_1^*.$$

**Proof.** For each equation and for $y, z \in X$ compute the $(y, z)$-entry of each side and interpret the results using Theorem 5.6, Corollary 6.3 and Lemma 6.5. □

**Lemma 7.3.** *With reference to Notation* 3.1 *and Definition* 7.1 *the following* (i)–(iii) *hold*:

(i) $FE_1^* = E_1^* A E_1^*$.
(ii) *For* $2 \leqslant i \leqslant D$

$$\begin{aligned} FE_i^* A_{i-1} E_1^* = {} & a_{i-1} E_i^* A_{i-1} E_1^* + (c_i - c_{i-1}) E_i^* A_{i-1} E_1^* A E_1^* \\ & + c_i (b^i - b^{i-2})(b^i - 1)^{-1} E_i^* (A_i - A_{i-1} E_1^* A) E_1^*. \end{aligned}$$

(iii) *For* $1 \leqslant i \leqslant D-1$

$$FE_i^* A_{i+1} E_1^* = a_i E_i^* A_{i+1} E_1^*.$$

**Proof.** For each equation and for $y, z \in X$ compute the $(y, z)$-entry of each side and interpret the results using Theorem 5.6, Corollary 6.3, Lemma 6.5 and Corollary 6.6. □

**Lemma 7.4.** *With reference to Notation* 3.1 *and Definition* 7.1 *the following* (i)–(iv) *hold*:

(i) *For* $1 \leqslant i \leqslant D-1$

$$RE_i^* A_{i-1} E_1^* = c_i E_{i+1}^* A_i E_1^*.$$

(ii) $RE_D^* A_{D-1} E_1^* = 0$.
(iii) *For* $1 \leqslant i \leqslant D-2$

$$\begin{aligned} RE_i^* A_{i+1} E_1^* = {} & c_{i+1} E_{i+1}^* A_{i+2} E_1^* + (c_{i+1} - c_i) E_{i+1}^* A_i E_1^* \\ & + (c_{i+1} - c_i) E_{i+1}^* A_i E_1^* A E_1^* \\ & + c_{i+1} (b^{i+1} - b^{i-1})(b^{i+1} - 1)^{-1} E_{i+1}^* (A_{i+1} - A_i E_1^* A) E_1^*. \end{aligned}$$

(iv)

$$\begin{aligned} RE_{D-1}^* A_D E_1^* = {} & (c_D - c_{D-1}) E_D^* A_{D-1} E_1^* + (c_D - c_{D-1}) E_D^* A_{D-1} E_1^* A E_1^* \\ & + c_D (b^D - b^{D-2})(b^D - 1)^{-1} E_D^* (A_D - A_{D-1} E_1^* A) E_1^*. \end{aligned}$$

**Proof.** For each equation and for $y, z \in X$ compute the $(y, z)$-entry of each side and interpret the results using Theorem 5.6, Corollary 6.3, Lemma 6.5 and Corollary 6.6. $\square$

## 8. More products in $T$

With reference to Notation 3.1, in this section we evaluate more products in $T$ which we will need later.

**Lemma 8.1.** *With reference to Notation 3.1, for $y, z \in \Gamma(x)$ and $1 \leqslant i \leqslant D$ the number $|\Gamma_i(x) \cap \Gamma_{i-1}(y) \cap \Gamma_{i-1}(z)|$ is equal to $c_i k_i k^{-1}$ if $y = z$, $0$ if $\partial(y, z) = 1$, and $c_i(c_i - 1)k_i k^{-1} b_1^{-1}$ if $\partial(y, z) = 2$.*

**Proof.** If $y = z$ then the result follows by Lemma 4.3(i). If $\partial(y, z) = 1$ then the result follows by Lemma 4.5(iii). Assume $\partial(y, z) = 2$. Abbreviate $D_j^\ell = D_j^\ell(x, z)$ $(0 \leqslant j, \ell \leqslant D)$ and note that $y \in D_2^1$. It follows from Theorem 5.6 that the number of paths of length $i - 1$ between $y$ and $D_{i-1}^i$ is independent of $y$. Moreover, between any two vertices of $\Gamma$ which are at distance $i - 1$, there exist exactly $c_1 c_2 \cdots c_{i-1}$ paths of length $i - 1$. Therefore, the scalar $|D_{i-1}^i \cap \Gamma_{i-1}(y)|$ is independent of $y$; denote this scalar by $\alpha_i$. For $v \in D_{i-1}^i$ we have $|\Gamma_{i-1}(v) \cap \Gamma(x)| = c_i$, so using Lemma 4.5(iii) we find $|\Gamma_{i-1}(v) \cap D_2^1| = c_i - 1$. Using these comments we count in two ways the number of pairs $(y, v)$ such that $y \in D_2^1$, $v \in D_{i-1}^i$, and $\partial(y, v) = i - 1$. This yields $\alpha_i |D_2^1| = |D_{i-1}^i|(c_i - 1)$. Evaluating this equation using Lemmas 4.2(i) and 4.3(i) we find $\alpha_i = c_i(c_i - 1)k_i k^{-1} b_1^{-1}$. The result follows. $\square$

**Corollary 8.2.** *With reference to Notation 3.1, for $1 \leqslant i \leqslant D$ we have*

$$E_1^* A_{i-1} E_i^* A_{i-1} E_1^* = c_i k_i k^{-1} E_1^* + c_i(c_i - 1)k_i k^{-1} b_1^{-1} E_1^* A_2 E_1^*.$$

**Proof.** For $y, z \in X$ we show that the $(y, z)$-entry of both sides are equal. If $y \notin \Gamma(x)$ or $z \notin \Gamma(x)$ then the $(y, z)$-entry of each side is 0. If $y, z \in \Gamma(x)$ then the $(y, z)$-entry of both sides are equal by Corollary 6.3, Lemmas 6.5 and 8.1. The result follows. $\square$

**Lemma 8.3.** *With reference to Notation 3.1, for $y, z \in \Gamma(x)$ and $1 \leqslant i \leqslant D - 1$ the number $|\Gamma_i(x) \cap \Gamma_{i-1}(y) \cap \Gamma_{i+1}(z)|$ is equal to $0$ if $y = z$, 0 if $\partial(y, z) = 1$, and $c_i b_i k_i k^{-1} b_1^{-1}$ if $\partial(y, z) = 2$.*

**Proof.** If $y = z$ then the result is clear. If $\partial(y, z) = 1$ then the result follows by Lemma 4.5(iii). Assume $\partial(y, z) = 2$. Abbreviate $D_j^\ell = D_j^\ell(x, z)$ $(0 \leqslant j, \ell \leqslant D)$ and note that $y \in D_2^1$. It follows from Theorem 5.6 that the number of paths of length $i - 1$ between $y$ and $D_{i+1}^i$ is independent of $y$. Moreover, between any two vertices of $\Gamma$ which are at distance $i - 1$, there exist exactly $c_1 c_2 \cdots c_{i-1}$ paths of length $i - 1$. Therefore, the scalar $|D_{i+1}^i \cap \Gamma_{i-1}(y)|$ is independent of $y$; denote this scalar by $\alpha_i$. For $v \in D_{i+1}^i$ we have $|\Gamma_{i-1}(v) \cap \Gamma(x)| = c_i$, so using Lemma 4.5(iii) we find $|\Gamma_{i-1}(v) \cap D_2^1| = c_i$. Using these comments we count in two ways the number of pairs $(y, v)$ such that $y \in D_2^1$, $v \in D_{i+1}^i$, and $\partial(y, v) = i - 1$. This yields $\alpha_i |D_2^1| = |D_{i+1}^i| c_i$. Evaluating this equation using Lemmas 4.2(i) and 4.3(i) and $c_{i+1} k_{i+1} = b_i k_i$ we find $\alpha_i = c_i b_i k_i k^{-1} b_1^{-1}$. The result follows. $\square$

**Corollary 8.4.** *With reference to Notation 3.1, for $1 \leqslant i \leqslant D-1$ we have*

$$E_1^* A_{i-1} E_i^* A_{i+1} E_1^* = c_i b_i k_i k^{-1} b_1^{-1} E_1^* A_2 E_1^*.$$

**Proof.** For $y, z \in X$ we show that the $(y, z)$-entry of both sides are equal. If $y \notin \Gamma(x)$ or $z \notin \Gamma(x)$ then the $(y, z)$-entry of each side is 0. If $y, z \in \Gamma(x)$ then the $(y, z)$-entry of both sides are equal by Corollary 6.3, Lemmas 6.5 and 8.3. The result follows. □

**Lemma 8.5.** *With reference to Notation 3.1, for $y, z \in \Gamma(x)$ and $1 \leqslant i \leqslant D-1$ the number $|\Gamma_i(x) \cap \Gamma_{i+1}(y) \cap \Gamma_{i+1}(z)|$ is equal to $b_i k_i k^{-1}$ if $y = z$, $b_i k_i k^{-1}$ if $\partial(y, z) = 1$, and $b_i(b_1 - a_i - c_i) k_i k^{-1} b_1^{-1}$ if $\partial(y, z) = 2$.*

**Proof.** If $y = z$ the result follows by Lemma 4.3(i) and since $c_{i+1}k_{i+1} = b_i k_i$. If $\partial(y, z) = 1$ then the result follows by Lemmas 4.3(i) and 4.5(iii) and since $c_{i+1}k_{i+1} = b_i k_i$. Assume $\partial(y, z) = 2$. Abbreviate $D_j^\ell = D_j^\ell(x, z)$ $(0 \leqslant j, \ell \leqslant D)$ and note that $y \in D_2^1$. We first claim that $|D_{i+1}^i \cap \Gamma_i(y)| = a_i b_i k_i k^{-1} b_1^{-1}$. It follows from Theorem 5.6 that the number of paths of length $i$ between $y$ and $D_{i+1}^i$ is independent of $y$. Moreover, between any two vertices of $\Gamma$ which are at distance $i-1$ ($i$, respectively), there exist exactly $(a_1 + a_2 + \cdots + a_{i-1}) c_1 c_2 \cdots c_{i-1}$ ($c_1 c_2 \cdots c_i$, respectively) paths of length $i$. By this and Lemma 8.3 the scalar $|D_{i+1}^i \cap \Gamma_i(y)|$ is independent of $y$; denote this scalar by $\alpha_i$. For $v \in D_{i+1}^i$ we have $|\Gamma_i(v) \cap \Gamma(x)| = a_i$, so using Lemma 4.5(iii) we find $|\Gamma_i(v) \cap D_2^1| = a_i$. Using these comments we count in two ways the number of pairs $(y, v)$ such that $y \in D_2^1$, $v \in D_{i+1}^i$, and $\partial(y, v) = i$. This yields $\alpha_i |D_2^1| = |D_{i+1}^i| a_i$. Evaluating this equation using Lemmas 4.2(i) and 4.3(i) and $c_{i+1}k_{i+1} = b_i k_i$ we find

$$\alpha_i = a_i b_i k_i k^{-1} b_1^{-1}. \tag{7}$$

We have proved the claim. We can now easily show that $|D_{i+1}^i \cap \Gamma_{i+1}(y)| = b_i(b_1 - a_i - c_i) k_i k^{-1} b_1^{-1}$. Pick $v \in D_{i+1}^i$. It follows from the triangle inequality that $\partial(y, v) \in \{i-1, i, i+1\}$, so

$$|D_{i+1}^i \cap \Gamma_{i+1}(y)| = |D_{i+1}^i| - |D_{i+1}^i \cap \Gamma_i(y)| - |D_{i+1}^i \cap \Gamma_{i-1}(y)|.$$

Using Lemmas 4.2(i), 4.3(i), 8.3 and (7) we find $|D_{i+1}^i \cap \Gamma_{i+1}(y)| = b_i(b_1 - a_i - c_i) k_i k^{-1} b_1^{-1}$. The result follows. □

**Corollary 8.6.** *With reference to Notation 3.1, for $1 \leqslant i \leqslant D-1$ we have*

$$E_1^* A_{i+1} E_i^* A_{i+1} E_1^* = b_i k_i k^{-1} E_1^* + b_i k_i k^{-1} E_1^* A E_1^* + b_i(b_1 - a_i - c_i) k_i k^{-1} b_1^{-1} E_1^* A_2 E_1^*.$$

**Proof.** For $y, z \in X$ we show that the $(y, z)$-entry of both sides are equal. If $y \notin \Gamma(x)$ or $z \notin \Gamma(x)$ then the $(y, z)$-entry of each side is 0. If $y, z \in \Gamma(x)$ then the $(y, z)$-entry of both sides are equal by Corollary 6.3, Lemmas 6.5 and 8.5. The result follows. □

## 9. Some scalar products

With reference to Notation 3.1, in this section we compute some scalar products which we will need later.

**Lemma 9.1.** *With reference to Notation* 3.1, *let $W$ denote an irreducible $T$-module with endpoint* 1. *Then $JW = 0$.*

**Proof.** Since $W$ is not the primary module we have $E_0W = 0$. Recall $J = |X|E_0$ so $JW = 0$. □

**Lemma 9.2.** *With reference to Notation* 3.1, *let $W$ denote an irreducible $T$-module with endpoint* 1. *Then the following* (i), (ii) *hold for $w \in E_1^*W$:*

(i) $E_i^*A_{i-1}w + E_i^*A_iw + E_i^*A_{i+1}w = 0$ *for* $1 \leqslant i \leqslant D-1$.
(ii) $E_D^*A_{D-1}w + E_D^*A_Dw = 0$.

**Proof.** For each equation in Corollary 6.4 apply both sides to $w$ and simplify using $E_1^*w = w$ and Lemma 9.1. □

**Corollary 9.3.** *With reference to Notation* 3.1, *let $W$ denote an irreducible $T$-module with endpoint* 1 *and local eigenvalue $\eta$. Then for $w \in E_1^*W$ we have $E_1^*A_2w = -(1+\eta)w$.*

**Proof.** Set $i = 1$ in Lemma 9.2(i) and note that $E_1^*A_0w = w$ and $E_1^*Aw = \eta w$. □

**Lemma 9.4.** *With reference to Notation* 3.1, *let $W$ denote an irreducible $T$-module with endpoint* 1 *and local eigenvalue $\eta$. Then the following* (i)–(iii) *hold for $w \in E_1^*W$:*

(i) $\|E_i^*A_{i-1}w\|^2 = (b_1 - (c_i-1)(1+\eta))c_ik_ik^{-1}b_1^{-1}\|w\|^2$ $(1 \leqslant i \leqslant D)$.
(ii) $\|E_i^*A_{i+1}w\|^2 = (k-b_i)(1+\eta)b_ik_ik^{-1}b_1^{-1}\|w\|^2$ $(1 \leqslant i \leqslant D-1)$.
(iii) $\langle E_i^*A_{i-1}w, E_i^*A_{i+1}w\rangle = -(1+\eta)c_ib_ik_ik^{-1}b_1^{-1}\|w\|^2$ $(1 \leqslant i \leqslant D-1)$.

**Proof.** (i) Evaluating $\|E_i^*A_{i-1}w\|^2 = \langle E_i^*A_{i-1}w, E_i^*A_{i-1}w\rangle$ using $E_1^*w = w$, line (1) and Corollary 8.2 we find

$$\|E_i^*A_{i-1}w\|^2 = \frac{c_ik_i}{k}\|w\|^2 + \frac{c_i(c_i-1)k_i}{kb_1}\langle w, E_1^*A_2w\rangle.$$

The result follows from this and Corollary 9.3.

(ii), (iii) Similar to the proof of (i) above. □

We now split the analysis into two cases, depending on whether $W$ has local eigenvalue $-1$ or $a_1$.

## 10. The irreducible $T$-modules with endpoint 1 and local eigenvalue $-1$

With reference to Notation 3.1, in this section we describe the irreducible $T$-modules with endpoint 1 and local eigenvalue $-1$.

**Lemma 10.1.** *With reference to Notation* 3.1, *let $W$ denote an irreducible $T$-module with endpoint* 1 *and local eigenvalue $-1$. Then for $w \in E_1^*W$ and $1 \leqslant i \leqslant D-1$ we have $E_i^*A_{i+1}w = 0$.*

**Proof.** Immediate from Lemma 9.4(ii). □

**Theorem 10.2.** *With reference to Notation 3.1, let $W$ denote an irreducible $T$-module with endpoint 1 and local eigenvalue $-1$. Fix a nonzero $w \in E_1^*W$. Then the following is a basis for $W$:*

$$E_i^* A_{i-1} w \quad (1 \leqslant i \leqslant D). \tag{8}$$

**Proof.** We first show that $W$ is spanned by the vectors (8). Let $W'$ denote the subspace of $V$ spanned by the vectors (8) and note that $W' \subseteq W$. We claim that $W'$ is a $T$-module. By construction $W'$ is $M^*$-invariant. It follows from Lemmas 7.2(i), (ii), 7.3(i), (ii), 7.4(i), (ii), 9.2 and 10.1, $E_1^* w = w$ and $E_1^* A w = -w$ that $W'$ is invariant under each of $L$, $F$, $R$. Recall that $L + F + R = A$ and $A$ generates $M$ so $W'$ is $M$-invariant. The claim follows. Note that $W' \neq 0$ since $w \in W'$ so $W' = W$ by the irreducibility of $W$. We now show that the vectors (8) are linearly independent. By (6) it suffices to show that $E_i^* A_{i-1} w \neq 0$ for $1 \leqslant i \leqslant D$. This follows from Lemma 9.4(i) and since $w \neq 0$. $\square$

**Corollary 10.3.** *With reference to Notation 3.1, let $W$ denote an irreducible $T$-module with endpoint 1 and local eigenvalue $-1$. Then $E_i^* W$ has dimension 1 for $1 \leqslant i \leqslant D$.*

**Proof.** Immediate from Theorem 10.2. $\square$

**Corollary 10.4.** *With reference to Notation 3.1, let $W$ denote an irreducible $T$-module with endpoint 1 and local eigenvalue $-1$. Then the following* (i), (ii) *hold*:

(i) *The dimension of $W$ is $D$.*
(ii) *The diameter of $W$ is $D - 1$.*

**Proof.** Immediate from Corollary 10.3. $\square$

**Corollary 10.5.** *With reference to Notation 3.1, let $W$ denote an irreducible $T$-module with endpoint 1 and local eigenvalue $-1$. Then $W = M^* M w$ for all nonzero $w \in E_1^* W$.*

**Proof.** By construction $M^* M w \subseteq W$ and equality holds in view of Theorem 10.2. $\square$

## 11. The irreducible $T$-modules with endpoint 1 and local eigenvalue $-1$: the $A$-action

With reference to Notation 3.1, let $W$ denote an irreducible $T$-module with endpoint 1 and local eigenvalue $-1$. In this section, we display the action of $A$ on the basis for $W$ given in Theorem 10.2. Since $A = L + F + R$ it suffices to give the actions of $L$, $F$, $R$ on this basis.

**Lemma 11.1.** *With reference to Notation 3.1, let $W$ denote an irreducible $T$-module with endpoint 1 and local eigenvalue $-1$. Then the following* (i), (ii) *hold for all nonzero $w \in E_1^* W$:*

(i) $Lw = 0$.
(ii) *For $2 \leqslant i \leqslant D$*

$$L E_i^* A_{i-1} w = b_{i-1} E_{i-1}^* A_{i-2} w.$$

**Proof.** For each equation of Lemma 7.2(i), (ii) apply each side to $w$ and simplify using $E_1^* w = w$ and Lemma 10.1. $\square$

**Lemma 11.2.** *With reference to Notation 3.1, let $W$ denote an irreducible $T$-module with endpoint 1 and local eigenvalue $-1$. Then the following holds for all nonzero $w \in E_1^* W$ and $1 \leqslant i \leqslant D$:*

$$F E_i^* A_{i-1} w = (a_{i-1} + c_{i-1} - c_i) E_i^* A_{i-1} w.$$

**Proof.** For each equation of Lemma 7.3(i), (ii) apply each side to $w$ and simplify using $E_1^* w = w$, $E_1^* A w = -w$, Lemmas 9.2 and 10.1. $\square$

**Lemma 11.3.** *With reference to Notation 3.1, let $W$ denote an irreducible $T$-module with endpoint 1 and local eigenvalue $-1$. Then the following* (i), (ii) *hold for all nonzero $w \in E_1^* W$:*

(i) *For $1 \leqslant i \leqslant D-1$*

$$R E_i^* A_{i-1} w = c_i E_{i+1}^* A_i w.$$

(ii) $R E_D^* A_{D-1} w = 0.$

**Proof.** For each equation of Lemma 7.4(i), (ii) apply each side to $w$ and simplify using $E_1^* w = w$. $\square$

## 12. The irreducible $T$-modules with endpoint 1 and local eigenvalue $a_1$

With reference to Notation 3.1, in this section we describe the irreducible $T$-modules with endpoint 1 and local eigenvalue $a_1$.

**Lemma 12.1.** *With reference to Notation 3.1, let $W$ denote an irreducible $T$-module with endpoint 1 and local eigenvalue $a_1$. For $w \in E_1^* W$ and $2 \leqslant i \leqslant D-1$ the determinant of*

$$\begin{pmatrix} \|E_i^* A_{i-1} w\|^2 & \langle E_i^* A_{i-1} w, E_i^* A_{i+1} w \rangle \\ \langle E_i^* A_{i+1} w, E_i^* A_{i-1} w \rangle & \|E_i^* A_{i+1} w\|^2 \end{pmatrix}$$

*is equal to*

$$c_i b_i (a_1 + 1)(a_i - a_1 c_i) k_i^2 k^{-1} b_1^{-2} \|w\|^4.$$

**Proof.** Evaluate the matrix entries using Lemma 9.4, take the determinant and simplify the result using (2). $\square$

**Theorem 12.2.** *With reference to Notation 3.1, let $W$ denote an irreducible $T$-module with endpoint 1 and local eigenvalue $a_1$. Fix a nonzero $w \in E_1^* W$. Then the following is a basis for $W$:*

$$E_i^* A_{i-1} w \quad (1 \leqslant i \leqslant D), \qquad E_i^* A_{i+1} w \quad (2 \leqslant i \leqslant D-1). \tag{9}$$

**Proof.** We first show that $W$ is spanned by the vectors (9). Let $W'$ denote the subspace of $V$ spanned by the vectors (9) and note that $W' \subseteq W$. We claim that $W'$ is a $T$-module. By construction $W'$ is $M^*$-invariant. It follows from Lemmas 7.2, 7.3, 7.4, 9.2, Corollary 9.3, $E_1^* w = w$ and

$E_1^*Aw = a_1w$ that $W'$ is invariant under each of $L$, $F$, $R$. Recall that $L + F + R = A$ and $A$ generates $M$ so $W'$ is $M$-invariant. The claim follows. Note that $W' \neq 0$ since $w \in W'$ so $W' = W$ by the irreducibility of $W$.

We now show that the vectors (9) are linearly independent. By (6) and since $w \neq 0$, it suffices to show that $E_i^*A_{i-1}w, E_i^*A_{i+1}$ are linearly independent for $2 \leqslant i \leqslant D-1$, and that $E_D^*A_{D-1}w \neq 0$. For $2 \leqslant i \leqslant D-1$ the vectors $E_i^*A_{i-1}w, E_i^*A_{i+1}w$ are linearly independent since their matrix of inner products has nonzero determinant by Corollary 3.2 and Lemma 12.1. It follows from (2) and Lemma 9.4(i) that $\|E_D^*A_{D-1}w\|^2 = (a_D - a_1c_D)c_Dk_Dk^{-1}b_1^{-1}\|w\|^2$. Now $E_D^*A_{D-1}w \neq 0$ by Corollary 3.2. By these comments the vectors (9) are linearly independent and the result follows. □

We emphasize an idea from the proof of Theorem 12.2.

**Corollary 12.3.** *With reference to Notation* 3.1, *let $W$ denote an irreducible $T$-module with endpoint* 1 *and local eigenvalue $a_1$. Fix a nonzero $w \in E_1^*W$. Then the following* (i)–(iii) *hold*:

(i) *$E_1^*W$ has a basis*

$$w.$$

(ii) *For $2 \leqslant i \leqslant D-1$ the subspace $E_i^*W$ has a basis*

$$E_i^*A_{i-1}w, \quad E_i^*A_{i+1}w.$$

(iii) *$E_D^*W$ has a basis*

$$E_D^*A_{D-1}w.$$

**Corollary 12.4.** *With reference to Notation* 3.1, *let $W$ denote an irreducible $T$-module with endpoint* 1 *and local eigenvalue $a_1$. Then the following* (i)–(iii) *hold*:

(i) *$E_1^*W$ has dimension* 1.
(ii) *$E_i^*W$ has dimension* 2 *for* $2 \leqslant i \leqslant D-1$.
(iii) *$E_D^*W$ has dimension* 1.

**Proof.** Immediate from Corollary 12.3. □

**Corollary 12.5.** *With reference to Notation* 3.1, *let $W$ denote an irreducible $T$-module with endpoint* 1 *and local eigenvalue $a_1$. Then the following* (i), (ii) *hold*:

(i) *The dimension of $W$ is $2D-2$.*
(ii) *The diameter of $W$ is $D-1$.*

**Proof.** Immediate from Corollary 12.4. □

**Corollary 12.6.** *With reference to Notation* 3.1, *let $W$ denote an irreducible $T$-module with endpoint* 1 *and local eigenvalue $a_1$. Then $W = M^*Mw$ for all nonzero $w \in E_1^*W$.*

**Proof.** By construction $M^*Mw \subseteq W$ and equality holds in view of Theorem 12.2. □

## 13. The irreducible $T$-modules with endpoint 1 and local eigenvalue $a_1$: the $A$-action

With reference to Notation 3.1, let $W$ denote irreducible $T$-module with endpoint 1 and local eigenvalue $a_1$. In this section, we display the action of $A$ on the basis for $W$ given in Theorem 12.2. Since $A = L + F + R$ it suffices to give the actions of $L$, $F$, $R$ on this basis.

**Lemma 13.1.** *With reference to Notation 3.1, let $W$ denote an irreducible $T$-module with endpoint 1 and local eigenvalue $a_1$. Then the following* (i)–(v) *hold for all nonzero $w \in E_1^* W$:*

(i) $Lw = 0$.
(ii) $LE_2^* Aw = (k - c_2(a_1 + 1))w$.
(iii) *For* $3 \leqslant i \leqslant D$

$$LE_i^* A_{i-1} w = b_{i-1} E_{i-1}^* A_{i-2} w + (c_i - c_{i-1}) E_{i-1}^* A_i w.$$

(iv) $LE_2^* A_3 w = -b_2(a_1 + 1)w$.
(v) *For* $3 \leqslant i \leqslant D - 1$

$$LE_i^* A_{i+1} w = b_i E_{i-1}^* A_i w.$$

**Proof.** For each equation of Lemma 7.2, apply each side to $w$ and simplify using $E_1^* w = w$ and Corollary 9.3. $\square$

**Lemma 13.2.** *With reference to Notation 3.1, let $W$ denote an irreducible $T$-module with endpoint 1 and local eigenvalue $a_1$. Then the following* (i)–(iv) *hold for all nonzero $w \in E_1^* W$:*

(i) $Fw = a_1 w$.
(ii) *For* $2 \leqslant i \leqslant D - 1$

$$\begin{aligned} FE_i^* A_{i-1} w = {} & (a_{i-1} + a_1(c_i - c_{i-1}) - c_i(a_1 + 1)(b^i - b^{i-2})(b^i - 1)^{-1}) E_i^* A_{i-1} w \\ & - c_i(b^i - b^{i-2})(b^i - 1)^{-1} E_i^* A_{i+1} w. \end{aligned}$$

(iii) $FE_D^* A_{D-1} w = (a_{D-1} + a_1(c_D - c_{D-1})$
$- c_D(a_1 + 1)(b^D - b^{D-2})(b^D - 1)^{-1}) E_D^* \cdot A_{D-1} w$.
(iv) *For* $2 \leqslant i \leqslant D - 1$

$$FE_i^* A_{i+1} w = a_i E_i^* A_{i+1} w.$$

**Proof.** For each equation of Lemma 7.3, apply each side to $w$ and simplify using $E_1^* w = w$, $E_1^* A w = a_1 w$ and Lemma 9.2. $\square$

**Lemma 13.3.** *With reference to Notation 3.1, let $W$ denote an irreducible $T$-module with endpoint 1 and local eigenvalue $a_1$. Then the following* (i)–(iv) *hold for all nonzero $w \in E_1^* W$:*

(i) *For* $1 \leqslant i \leqslant D - 1$

$$RE_i^* A_{i-1} w = c_i E_{i+1}^* A_i w.$$

(ii) $RE_D^* A_{D-1} w = 0$.

(iii) *For* $2 \leqslant i \leqslant D-2$

$$RE_i^* A_{i+1} w = (a_1+1)(c_{i+1}(b^{i-1}-1)(b^{i+1}-1)^{-1} - c_i)E_{i+1}^* A_i w + c_{i+1}(b^{i-1}-1)(b^{i+1}-1)^{-1} E_{i+1}^* A_{i+2} w.$$

(iv) $RE_{D-1}^* A_D w = (a_1+1)(c_D(b^{D-2}-1)(b^D-1)^{-1} - c_{D-1})E_D^* A_{D-1} w.$

**Proof.** For each equation of Lemma 7.4, apply each side to $w$ and simplify using $E_1^* w = w$, $E_1^* A w = a_1 w$ and Lemma 9.2. □

## 14. The isomorphism class of an irreducible $T$-module with endpoint 1

With reference to Notation 3.1, in this section we prove that up to isomorphism there exist exactly two irreducible $T$-modules with endpoint 1.

**Proposition 14.1.** *With reference to Notation* 3.1, *any two irreducible $T$-modules with endpoint* 1 *and local eigenvalue* $-1$ *are isomorphic.*

**Proof.** Let $W$ and $W'$ denote irreducible $T$-modules with endpoint 1 and local eigenvalue $-1$. Fix nonzero $w \in E_1^* W$, $w' \in E_1^* W'$. By Theorem 10.2, $W$ and $W'$ have bases $\{E_i^* A_{i-1} w \mid 1 \leqslant i \leqslant D\}$ and $\{E_i^* A_{i-1} w' \mid 1 \leqslant i \leqslant D\}$, respectively. Let $\sigma : W \to W'$ denote the vector space isomorphism defined by $\sigma(E_i^* A_{i-1} w) = E_i^* A_{i-1} w'$ for $1 \leqslant i \leqslant D$. We show that $\sigma$ is a $T$-module isomorphism. Since $A$ generates $M$ and $E_0^*, E_1^*, \ldots, E_D^*$ is a basis for $M^*$, it suffices to show that $\sigma$ commutes with each of $A, E_0^*, E_1^*, \ldots, E_D^*$.

Using the assertion (iv) below the line (5) and the definition of $\sigma$ we immediately find that $\sigma$ commutes with each of $E_0^*, E_1^*, \ldots, E_D^*$. It follows from Lemmas 11.1–11.3 that $\sigma$ commutes with each of $L, F, R$. Recall $A = L + F + R$ so $\sigma$ commutes with $A$. The result follows. □

**Proposition 14.2.** *With reference to Notation* 3.1, *any two irreducible $T$-modules with endpoint* 1 *and local eigenvalue* $a_1$ *are isomorphic.*

**Proof.** Similar to the proof of Proposition 14.1. □

**Corollary 14.3.** *With reference to Notation* 3.1 *fix a nonzero* $w \in E_1^* V$ *which is orthogonal to* $\sum_{y \in \Gamma(x)} \hat{y}$. *Assume that* $w$ *is an eigenvector for* $E_1^* A E_1^*$. *Then* $M^* M w$ *is an irreducible $T$-module with endpoint* 1.

**Proof.** Let $H$ denote the subspace of $V$ spanned by the irreducible $T$-modules with endpoint 1. By construction and Lemma 9.1 $E_1^* H$ is the orthogonal complement of $\sum_{y \in \Gamma(x)} \hat{y}$ in $E_1^* V$. Hence $w \in E_1^* H$. Note that $Tw \subseteq H$ so $Tw$ is the orthogonal direct sum of some irreducible $T$-modules of endpoint 1. Call these $T$-modules $W_1, W_2, \ldots, W_s$. We show $s = 1$. By construction and since $w \in E_1^* V$ there exist $w_i \in E_1^* W_i$ $(1 \leqslant i \leqslant s)$ such that

$$w = w_1 + w_2 + \cdots + w_s. \tag{10}$$

For $1 \leqslant i \leqslant s$ we have $w_i \neq 0$; otherwise $Tw \subseteq W_1 + \cdots + W_{i-1} + W_{i+1} + \cdots + W_s$. We claim that the $T$-modules $W_1, W_2, \ldots, W_s$ are mutually isomorphic. To see this, recall that $w$ is an eigenvector for $E_1^* A E_1^*$; let $\eta$ denote the corresponding eigenvalue. Applying $E_1^* A E_1^*$ to each term in (10) and using $E_1^*AE_1^* \in T$ we find $E_1^*AE_1^*w_i = \eta w_i$ for $1 \leqslant i \leqslant s$. Therefore, each of $W_1, W_2, \ldots, W_s$ has local eigenvalue $\eta$, so $W_1, W_2, \ldots, W_s$ are mutually isomorphic by Propositions 14.1 and 14.2. We have proved the claim. We can now easily show that $s = 1$. Suppose $s \geqslant 2$. By construction there exists $t \in T$ such that $tw = w_1$. We have $w_1 = tw = tw_1 + \cdots + tw_s$ and $tw_i \in W_i$ for $1 \leqslant i \leqslant s$. Therefore, $tw_i = 0$ for $2 \leqslant i \leqslant s$. Now $(t - I)E_1^*$ is zero on $W_1$ and nonzero on $W_i$ for $2 \leqslant i \leqslant s$; this contradicts the fact that $W_1, W_2, \ldots, W_s$ are mutually isomorphic. We conclude $s = 1$. Now $Tw = W_1$ is an irreducible $T$-module with endpoint 1. The result follows since $Tw = M^*Mw$ by Corollaries 10.5 and 12.6. □

With reference to Notation 3.1 recall that $V$ is an orthogonal direct sum of irreducible $T$-modules. Let $W$ denote an irreducible $T$-module. By the *multiplicity with which $W$ appears in $V$* we mean the number of irreducible $T$-modules in this sum which are isomorphic to $W$. For example the primary module $M\hat{x}$ appears in $V$ with multiplicity 1.

**Theorem 14.4.** *With reference to Notation 3.1, up to isomorphism there exist exactly two irreducible $T$-modules with endpoint 1. The first has local eigenvalue $-1$ and appears in $V$ with multiplicity $ka_1(a_1+1)^{-1}$. The second has local eigenvalue $a_1$ and appears in $V$ with multiplicity $b_1(a_1+1)^{-1}$.*

**Proof.** By Corollary 4.7 each irreducible $T$-module with endpoint 1 has local eigenvalue $-1$ or $a_1$. By Proposition 14.1 (resp. Proposition 14.2) any two irreducible $T$-modules with endpoint 1 and local eigenvalue $-1$ (resp. $a_1$) are isomorphic. For $\eta \in \{a_1, -1\}$ let $\mu_\eta$ denote the multiplicity with which an irreducible $T$-module with endpoint 1 and local eigenvalue $\eta$ appears in $V$. We show that $\mu_\eta = ka_1(a_1+1)^{-1}$ if $\eta = -1$ and $\mu_\eta = b_1(a_1+1)^{-1}$ if $\eta = a_1$. Let $H_\eta$ denote the subspace of $V$ spanned by all the irreducible $T$-modules with endpoint 1 and local eigenvalue $\eta$. We claim that $\mu_\eta$ is equal to the dimension of $E_1^*H_\eta$. Observe that $H_\eta$ is a $T$-module so it is an orthogonal direct sum of irreducible $T$-modules:

$$H_\eta = W_1 + W_2 + \cdots + W_m \quad \text{(orthogonal direct sum)}, \tag{11}$$

where $m$ is a nonnegative integer, and where $W_1, W_2, \ldots, W_m$ are irreducible $T$-modules with endpoint 1 and local eigenvalue $\eta$. Apparently $m$ is equal to $\mu_\eta$. We show $m$ is equal to the dimension of $E_1^*H_\eta$. Applying $E_1^*$ to (11) we find

$$E_1^*H_\eta = E_1^*W_1 + E_1^*W_2 + \cdots + E_1^*W_m \quad \text{(direct sum)}. \tag{12}$$

Note that each summand on the right in (12) has dimension 1. It follows that $m$ is equal to the dimension of $E_1^*H_\eta$ and the claim is proven. Recall that $\sum_{y\in\Gamma(x)}\hat{y}$ is an eigenvector for $E_1^*AE_1^*$ with eigenvalue $a_1$. Let $U_\eta$ denote the set of those vectors in $E_1^*V$ that are eigenvectors for $E_1^*AE_1^*$ with eigenvalue $\eta$ and that are orthogonal to $\sum_{y\in\Gamma(x)}\hat{y}$. By Corollary 4.6(ii) the dimension of $U_\eta$ is $ka_1(a_1+1)^{-1}$ if $\eta = -1$ and $k(a_1+1)^{-1} - 1 = b_1(a_1+1)^{-1}$ if $\eta = a_1$. We now show $E_1^*H_\eta = U_\eta$. By (12) and Lemma 9.1 we find $E_1^*H_\eta \subseteq U_\eta$. Pick a nonzero $w \in U_\eta$. By Corollary 14.3 and definition of $H_\eta$ we find $w \in E_1^*M^*Mw \subseteq E_1^*H_\eta$ implying $U_\eta \subseteq E_1^*H_\eta$. It follows $U_\eta = E_1^*H_\eta$. The result now follows from these comments and the above claim. □

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